# Landau Damping and Coherent Structures in Narrow-Banded 1+1 Deep Water Gravity Waves

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We study the nonlinear energy transfer around the peak of the spectrum of surface gravity waves by taking into account nonhomogeneous effects. In the narrow-banded approximation the kinetic equation resulting from a nonhomogeneous wave field is a Vlasov-Poisson type equation which includes at the same time the random version of the Benjamin-Feir instability and the Landau damping phenomenon. We analytically derive the values of the Phillips' constant $\alpha$ and the enhancement factor $\gamma$ for which the narrow-banded approximation of the JONSWAP spectrum is unstable. By performing numerical simulations of the nonlinear Schrödinger equation we check the validity of the prediction of the related kinetic equation. We find that the effect of Landau damping is to suppress the formation of coherent structures. The problem of predicting freak waves is briefly discussed.

In many different fields of nonlinear physics, local nonlinear effects such as the modulational instability (MI) have played a very important role in spectral energy transfer processes (see for example [1–3]). For ocean gravity waves, the same instability (now commonly known as the Benjamin-Feir instability) has been discovered independently by Benjamin and Feir [4] and by Zakharov [5] in the sixties. The instability predicts that in deep water a monochromatic wave is unstable under suitable small perturbations. In this framework it has been established that the MI can be responsible for the formation of freak waves [6–8]. The aim of this Letter is to discuss a statistical approach that allows one to predict the appearance of the modulational instability in a continuous spectrum. The theory is applied directly to ocean waves but it could as well be applied to many other nonlinear wave systems.

In order to approach statistically the nonlinear energy transfer processes involved in the MI, one is interested in finding a suitable kinetic description. As far as ocean waves are concerned, today the most common models for wave forecasting (WAM, SWAM and newer generation models, see [9]) are based on the nonlinear energy transfer process that is ruled by the *kinetic wave equation* that has been derived independently by K. Hasselmann [10] and by V. Zakharov [5]. The theory predicts that the energy is transferred in an irreversible manner under the resonant 4-wave interaction. Besides the quasi-gaussian approximation [11], one of the major hypothesis required to derive this kinetic equation is the homogeneity of the surface, i.e., $< A(k)A^*(k') >= n(k)\delta(k-k')$, where $A$ is a complex wave amplitude describing the evolution of the wave train, $k$ and $k'$ are wave numbers and $n(k)$ is the spectral density function. Unfortunately, the Hasselmann-Zakharov kinetic theory is not able to predict the Benjamin-Feir instability. This basically means that the kinetic equation is unable to predict local nonlinear effects such as freak waves. Indeed the MI is the result of an interaction of waves that are phase-locked: the carrier wave is phase-locked with the side-bands and therefore the mechanism cannot be predicted by the assumption that the Fourier components are delta-correlated. Thus, if the hypothesis of homogeneity of the system is relaxed, an improved kinetic equation can be derived which is able to show a random version of the Benjamin-Feir instability. Actually, for surface gravity waves, this improvement is contained in the pioneering work by Alber [12], followed by the works of Crawford et al. [13] and Janssen [14,15]. The prediction of the theory developed in [12–14] has never been verified numerically or experimentally. Unfortunately these works, written more than 15 years ago, did not receive the deserved attention by the wave forecast/hindcast community and the ideas developed remained basically unconsidered with the only exception of the recent work by M. Stiassnie [16]. More recently, but independently of the works [12–14], a similar approach has been developed for the large-amplitude electromagnetic wave-envelope propagation in nonlinear media [17], for the quantum-like description of the longitudinal charged-particle beam dynamics in high-energy accelerating machines in the presence of an arbitrary coupling impedance [18] and for the resonant interaction between an instantaneously-produced disturbance and a partially incoherent Langmuir wave [19].

In this paper, we outline the approach formulated in [12–14] and discuss the modulational instability for random wave spectra. In particular we identify the values of the parameters of the JONSWAP spectrum for which the spectrum itself is unstable. Moreover, using numerical simulations we associate the MI with the presence of coherent structures in physical space, pointing out the importance of the role played by the phenomenon of Landau damping.

We start by considering the simplest and prototypical

weakly nonlinear equation for water waves: the Nonlinear Schroedinger equation (NLSE). This choice is dictated by the following motivations. First of all, we want to avoid dealing with the dynamics of the four-wave resonant interactions because otherwise we would not be able to discern whether the effects we are describing are due to inhomogeneity or are already contained in the Hasselmann-Zakharov theory. We point out that, while the two dimensional NLSE (and higher order equations (see [20])) contains the four-wave resonant interaction mechanism [21], the one-dimensional NLSE does not include this sort of energy transfer, because of the shape of the linear dispersion relation (see for instance [22]). Second, while the NLSE should not be appropriate for describing the dynamics in the tail of the spectrum or in the inertial range, it should describe with satisfactory accuracy the short time scale behavior around the peak. Formally the NLSE can be derived from the Zakharov equation [5] under the narrow band approximation; in dimensional form the equation reads:

$$\frac{\partial A}{\partial t} + i\mu\frac{\partial^2 A}{\partial x^2} + i\nu|A|^2 A = 0, \tag{1}$$

where in deep water $\mu = \omega_0/8k_0^2$ and $\nu = \omega_0 k_0^2/2$, with $\omega_0$ the carrier angular frequency and $k_0$ the respective wave number.

Eq. (1) is our starting point for deriving the required kinetic equation. Following Alber [12], the Wigner-Moyal transform [23] can be applied directly to the NLSE. This transform allows one to give a representation of a function $A(x)$ both in configuration space, $x$, and in phase space or wave number, $k$, namely

$$n(x,k) = \frac{1}{2\pi}\int < A^*(x+y/2)A(x-y/2) > e^{-iky}dy. \tag{2}$$

Alternatively, one may follow the approach presented both in [13] and in [24] consisting in writing the NLSE in Fourier space and then writing an evolution equation for the correlator $n(k,k') =< A(k)A^*(k') >$. Both approaches need a closure in order to reduce the fourth-order correlator into the sum of the product of the two-point correlation functions. Using any of the methods outlined the resulting kinetic equation is the following von Neumann-Weyl-like equation

$$\frac{\partial n(x,k,t)}{\partial t} + 2\mu k\frac{\partial n(x,k,t)}{\partial x} + 4\nu\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!2^{2m+1}}\times \frac{\partial^{2m+1} < |A(x,t)|^2 >}{\partial x^{2m+1}}\frac{\partial^{2m+1} n(x,k,t)}{\partial k^{2m+1}} = 0, \tag{3}$$

with

$$< |A(x,t)|^2 >= \int n(x,k,t)dk. \tag{4}$$

If the limit for small $k$ is taken the equation resembles the Vlasov-Poisson equation in plasma physics that is well known to describe the Landau damping phenomenon. The standard way to proceed consists in letting the distribution function $n(x,k,t)$ and the field $A(x,t)$ be expressed in terms of an equilibrium value plus a small perturbation and study the dispersion relation of the linearized equation for the perturbation. After standard algebra the following dispersion relation is obtained (see also [17–19]):

$$1 + \frac{\nu}{\mu}\int\frac{n_0(k+K/2) - n_0(k-K/2)}{K(k-\Omega/(2\mu K))}dk = 0, \tag{5}$$

where $n_0$ is the homogeneous envelope spectrum.

According to experimental studies, the surface wave spectrum for gravity waves is given by the JONSWAP spectrum:

$$P(k) = \frac{\alpha}{2k^3}e^{-\frac{3}{2}[\frac{k_0}{k}]^2}\gamma^{exp[-\frac{(\sqrt(k)-\sqrt(k_0))^2}{2\delta^2 k_0}]}, \tag{6}$$

with $\alpha$, $\gamma$ and $\delta$ constants ($\delta$ is usually set to 0.07, while $\alpha$ and $\gamma$ depend on the state of the ocean). In order to be consistent with the NLSE, as is done in [15], we Taylor-expand the spectrum around its peak and obtain the following Lorenzian spectrum:

$$P(k) = \frac{H_s^2}{16\pi}\frac{p}{p^2+(k-k_0)^2}, \tag{7}$$

where

$$p = \sqrt{\frac{8k_0^2\delta^2}{24\delta^2 + Log(\gamma)}} \text{ and } H_s = 4\sqrt{\pi\frac{\alpha\gamma p}{2E^{3/2}k_0^3}}. \tag{8}$$

$H_s$ is the significant wave height calculated as four times the standard deviation of the wave field and $p$ corresponds exactly to the half-width at half-maximum of the spectrum. It can be shown [13] that for a symmetric spectrum $P(k)$ of the surface elevation, the spectrum for the complex envelope is $n_0(k) = 4P(k-k_0)$, therefore a factor of four must be taken into account. Substituting in (5) we obtain the following dispersion relation:

$$\Omega = K(\sqrt{K^2\mu^2 - H_s^2\nu\mu} - 2i\mu p). \tag{9}$$

If $K^2 < H_s^2\nu/\mu$, the first term on the right hand side is responsible for the MI (note that in the limit as $p \to 0$, the dispersion relation (9) gives the Benjamin-Feir instability). The last term on the right hand side is responsible for the Landau damping phenomenon [25]. Therefore there is a competition between exponential growth and damping of the perturbation that depends on the parameters $\alpha$ and $\gamma$ of the Lorenzian (or JONSWAP) spectrum. If $p > \sqrt{-K^2\mu/4 + H_s^2\nu/4}$ the damping dominates the MI, the opposite will occur if $p < \sqrt{-K^2\mu/4 + H_s^2\nu/4}$. In Fig. 1 we show the instability diagram in the $\alpha - \gamma$ plane. In general spectra with higher values of $\alpha$ and $\gamma$

are more likely to show the MI. This results gives support to the results in [8] where it has been found that higher values of $\alpha$ and $\gamma$ increases the probability for the formation of freak waves.

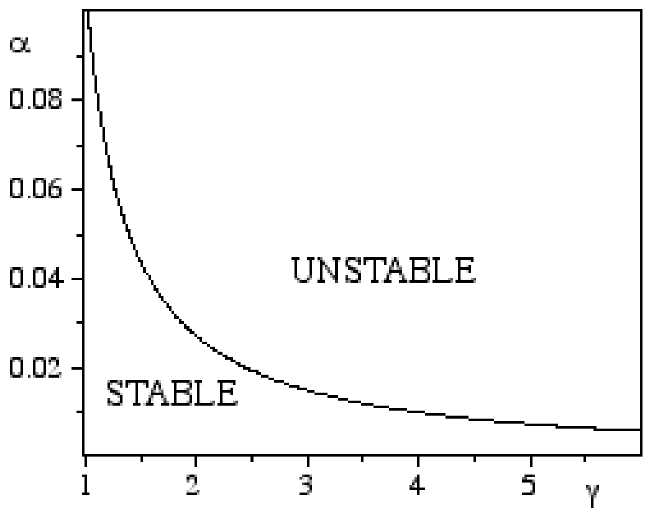


FIG. 1. Instability diagram in the $\alpha - \gamma$ plane.

Before comparing numerical simulations with the prediction of the Wigner-Moyal-like kinetic equation, we would like to point out that our analysis has been carried out linearizing around an equilibrium state of the system. In fact, the dispersion relation (9) has been derived for small amplitude perturbations. However, in the natural long-time evolution of a nonlinear wave train, perturbations are in general not small. Consequently, perturbations in the simulations should evolve according to the governing equations. To this end, numerical experiments have been carried out allowing the nonhomogeneities to develop by themselves according to the nonlinear dynamics of the NLSE.

Numerical simulations of Eq. (1) have been computed using a standard pseudo-spectral Fourier method. Initial conditions for the free surface elevation $\zeta(x,0)$ have been constructed as a random process [26], i.e. a linear superposition of Fourier modes (6) with random phases. The Hilbert transform is used in order to convert the free surface $\zeta$ to the complex envelope variable $A$ of the NLSE. The dominant wave number for the numerical simulation was selected to be $k_0 = 0.1\ m^{-1}$. This last choice is not restrictive: the parameters that rule the dynamics in the spectrum are its width and the steepness which, once $k_0$ is fixed, are univocally determined by $\alpha$ and $\gamma$.

We start the discussion on the numerical results by showing the evolution of $|A(x,t)|$ in the $x - t$ plane for a case in which $\gamma = 3$ and $\alpha = 0.02$ ($p/k_0 \simeq 0.25$ and $H_s k_0/2 \simeq 0.14$ ), Fig. 2. According to Fig. 1 the spectrum should be unstable. How is this instability manifested in configuration space? From Fig. 2 we note the presence of a "coherent structure", i.e. a structure (oblique darker zones in the $x - t$ plane) that persists in the presence of nonlinear interactions and maintains statistically its shape and velocity during propagation (note that periodic boundary conditions are used). The presence of such coherent structures is related to the values of $\alpha$ and $\gamma$ and not to the particular random phases selected.

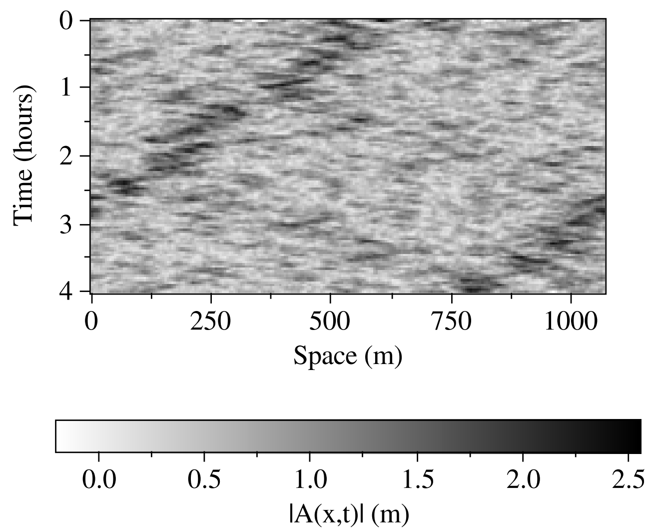


FIG. 2. $|A(x,t)|$ from numerical simulation of the NLSE. The initial condition is characterized by a Lorentian spectrum with $\gamma = 3$, $\alpha = 0.02$. A coherent structure is evident in the $x - t$ plane.

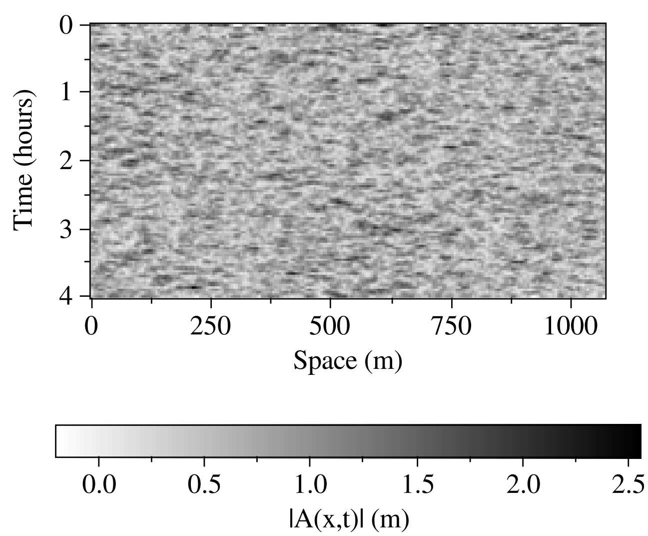


FIG. 3. $|A(x,t)|$ from numerical simulation of the NLSE. The initial condition is characterized by a Lorentian spectrum with $\gamma = 1$, $\alpha = 0.02$. The field in the $x - t$ plane appear to be random with out any evidence of coherent structure.

Every random realization shows similar results even though the resulting coherent structures may have different velocity and amplitude. The nonlinear stage of MI is therefore responsible for the formation of such coherent structures in the $x - t$ plane. Indeed it is possible to show that the NLSE has periodic solutions such as, for exam-

ple, breathers or unstable modes [27,28]. These solutions, which are linearly unstable, are nevertheless very robust. Moreover they can grow up to more than three times the initial unperturbed solution and therefore have also been addressed as simple models for freak waves [28,29]. In contrast to solitons that have constant amplitude in time, these unstable modes are characterized by a continuous exchange of energy among the Fourier modes. The energy is transferred from one mode to another and back again: the process is completely reversible and therefore coherent structures persist in physical space. We stress that these kinds of solutions appear naturally from initial conditions with random phases. We now consider a case with $\gamma = 1$ and $\alpha = 0.02$ ($p/k_0 \simeq 0.58$ and $H_s k_0/2 \simeq 0.12$); note that decreasing $\gamma$ has the effect of making the spectrum broader. This case, according to the stability criteria (see Fig. 1), should not present any instability because of the effect of Landau damping. We show the output of our numerical simulation in Fig 3. In the $x-t$ plane, as a confirmation of what we have previously stated, there is no evidence of any coherent structure; numerical simulations with different random phases are in accordance with the result just shown.

In conclusion, the numerical simulations show that the presence of the coherent structures in the $x-t$ plane is related to the instability of the wave field. The Landau damping phenomenon suppress the instability and prevent the formation of coherent structures. Many physical questions remain open. For example it would be interesting to investigate the case of a two dimensional wave field. It is well known that the NLSE in 2+1 is not integrable and the dynamics of coherent structures is still far from being understood. Numerical simulations with the fully nonlinear Euler equations are also under consideration in order to extend the validity of the result. Concerning the problem of wave forecasting, we may state that if one is interested in predicting freak waves arising from the MI, this new form of the kinetic equation, which includes nonhomogeneous effects should be considered.

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